Metric Spaces

22/1/2020

DEFINITION: Let X be any non-empty set. Then a metric on X, denoted by ρ, is a real valued function $\rho : X \times X \to \mathbb{R}$ that satisfies the following conditions:

M1: $\rho(x, x) = 0 \iff x = 0 \ \forall x \in X$ (nullity)

M2: $\rho(x, y) \geq 0, \ \forall x, y \in X$ (non-negative)

M3: $\rho(x, y) = \rho(y, x), \ \forall x, y \in X$. (symmetric)

M4: $\rho(x, y) \leq \rho(x, z) + \rho(z, y), \ \forall x, y, z \in X$ (tri-angular Inequal)

The pair (X, ρ) consisting of a non-empty set X and a metric ρ on X, is called a metric space.

Example 1: Let $X = \mathbb{R}$ and let ρ be defined by $\rho(x, y) = |x - y|$. Then $(\mathbb{R}, |\cdot|)$ is a metric space ~~bec~~ $|\cdot|$ is a metric on $\mathbb{R}$.

Varification:

M1: $\rho(x, x) = |x - x| = 0 \iff x = 0 \ \forall x \in X$

M2: $\rho(x, y) = |x - y| \geq 0, \ \forall x, y \in X$

M3: $\rho(x, y) = |x - y| = |y - x| = \rho(y, x) \ \forall x, y \in X$.

M4: $\rho(x, y) = |x - y| = |x - z + z - y|$

$$\leq |x - z| + |z - y|$$
$$= \rho(x, z) + \rho(z, y)$$

$\Rightarrow \rho(x, y) \leq \rho(x, z) + \rho(z, y) \ \forall x, y, z \in X$.

<u>Remark:</u> $\rho(x, x) = 0 \iff x = 0$ mean that $\rho(x, y) = 0 \iff x = y$ so. hence from now we shall be using the condition.

Example 2: Let $X = \mathbb{R}^2$ and let ρ be defined on $\mathbb{R}^2$
$\rho(x,y) = \max\{|x_i - y_i|\}$ for $1 \leq i \leq 2$, then ρ is a metric on $\mathbb{R}^2$ and so $(\mathbb{R}^2, \rho)$ is a metric space.

varification

M1: $\rho(x,y) = \max\{|x_i - y_i|\} = 0$ for $1 \leq i \leq 2$

$\iff |x_i - y_i| = 0$

$\iff x_i - y_i = 0$

$\iff x_i = y_i$

$\iff x = y$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$

M2: $\rho(x,y) = \max\{|x_i - y_i|\} \geq 0$ being $1 \leq i \leq 2$

the maximum of non-negative numbers.

M3: $\rho(x,y) = \max\{|x_i - y_i|\}$ $1 \leq i \leq 2$

$= \max\{|y_i - x_i|\} = \rho(y,x)$ $1 \leq i \leq 2$

M4: $\rho(x,y) = \max\{|x_i - y_i|\}$ $1 \leq i \leq 2$

$= \max\{|x_i - z_i + z_i - y_i|\}$ $1 \leq i \leq 2$

$\leq \max\{|x_i - z_i|\} + \{|z_i - y_i|\}$ $1 \leq i \leq 2$

$= \rho(x,z) + \rho(z,y)$.

Hence ρ is a metric space on $\mathbb{R}^2$ $\square$

Example 3: Let $X = \mathbb{R}^2$ and let ρ be defined on $\mathbb{R}^2$ by

$\rho(x,y) = \sum_{i=1}^{2} |x_i - y_i|$. Then $(\mathbb{R}^2, \rho)$ is a metric space i.e. ρ is a metric space on $\mathbb{R}^2$.

Varification

M1: $\rho(x,y) = \sum_{i=1}^{2} |x_i - y_i| = 0$

$\Leftrightarrow |x_i - y_i| = 0, \ \forall i = 1,2$

$\Leftrightarrow x_i - y_i = 0, \ \forall i = 1,2$

$\Leftrightarrow x_i = y_i, \ \forall i = 1,2$

$\Leftrightarrow x = y$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $\mathbb{R}^2$.

M2: $\rho(x,y) = \sum_{i=1}^{2} |x_i - y_i| \geq 0$

Being the finite sum of non negative numbers.

M3: $\rho(x,y) = \sum_{i=1}^{2} |x_i - y_i|$

$= \sum_{i=1}^{2} |y_i - x_i| = \rho(y,x)$

M4: $\rho(x,y) = \sum_{i=1}^{2} |x_i - y_i|$

$= \sum_{i=1}^{2} |x_i - z_i + z_i - y_i|$

$\leq \sum_{i=1}^{2} |x_i - z_i| + \sum_{i=1}^{2} |z_i - y_i|$

$= \rho(x,z) + \rho(z,y)$

Hence ρ is a metric space on $\mathbb{R}^2$ $\square$.

xample 4: Let $C([a,b])$ be the set of all Constants real valued function defined on $[a,b]$. Then the mapping ρ define on $[a,b]$ by

$$\rho(f,g) = \sup_{t \in [a,b]} \{|f(t) - g(t)|\}$$ is a metric and $(C([a,b]), \rho)$ is a metric space.

varification

M1: $\rho(f,g) = \sup_{t\in[a,b]} \{|f(t) - g(t)|\} = 0$

$\Leftrightarrow |f(t) - g(t)| = 0, \quad t \in [a,b]$

$\Leftrightarrow f(t) - g(t) = 0, \quad t \in [a,b]$

$\Leftrightarrow f(t) = g(t), \quad t \in [a,b]$

$\Leftrightarrow f = g$

M2: $\rho(f,g) = \sup_{t\in[a,b]} \{|f(t) - g(t)|\} \geq 0$

being the supremum of non-negative numbers.

M3: $\rho(f,g) = \sup_{t\in[a,b]} \{|f(t) - g(t)|\}$

$= \sup_{t\in[a,b]} \{|g(t) - f(t)|\} = \rho(g,f)$

M4: $\rho(f,g) = \sup_{t\in[a,b]} \{|f(t) - g(t)|\}$

$= \sup_{t\in[a,b]} \{|f(t) - h(t) + h(t) - g(t)|\}$

$\leq \sup_{t\in[a,b]} \{|f(t) - h(t)|\} + \{|h(t) - g(t)|\}$

$= \rho(f,h) + \rho(h,g)$

Hence it is a metric space $\square$

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example 5: Let $B([a,b])$ be the set of all bounded real valued function on $[a,b]$. Then the mapping ρ define on $[a,b]$ by $\rho(f,g) = \sup_{t\in[a,b]} \{|f(t)-g(t)|\}$ is a metric and $([a,b],\rho)$ is a metric space.

verification:

M1: $\rho(f,g) = \sup_{t\in[a,b]} \{|f(t)-g(t)|\} = 0$

$\iff |f(t)-g(t)| = 0, \quad t\in[a,b]$

$\iff f(t)-g(t) = 0, \quad t\in[a,b]$

$\iff f(t) = g(t), \quad t\in[a,b]$

$\iff f = g$

M2: $\rho(f,g) = \sup_{t\in[a,b]} \{|f(t)-g(t)|\} \geq 0$

Being the supremum of non-negative numbers.

M3: $\rho(f,g) = \sup_{t\in[a,b]} \{|f(t)-g(t)|\}$

$= \sup_{t\in[a,b]} \{|g(t)-f(t)|\} = \rho(g,f)$.

M4: $\rho(f,g) = \sup_{t\in[a,b]} \{|f(t)-g(t)|\}$

$= \sup_{t\in[a,b]} \{|f(t)-k(t)+k(t)-g(t)|\}$

$\leq \sup_{t\in[a,b]} \{|f(t)-k(t)|\} + \sup_{t\in[a,b]} \{|k(t)-g(t)|\}$

$= \rho(f,k) + \rho(k,g) \qquad \square$

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Example 6: Let C be the set of all convergent sequences of real nos and let ρ be a function defined on C by $\rho(x,y) = \sup\{|x_i - y_i| : i \in N\}$. Then ρ is a metric on C and so (C, ρ) is a metric space.

varification

M1: $\rho(x,y) = \sup\{|x_i - y_i|\} = 0$

$\iff |x_i - y_i| = 0 ; \quad i \in N$

$\iff x_i - y_i = 0 , \quad i \in N$

$\iff x_i = y_i , \quad i \in N$

$\iff x = y$

M2: $\rho(x,y) = \sup\{|x_i - y_i|\} \geq 0$

being the supremum of non-negative numbers.

M3: $\rho(x,y) = \sup\{|x_i - y_i|\}$

$= \sup\{|y_i - x_i|\} = \rho(y,x)$

M4: $\rho(x,y) = \sup\{|x_i - y_i|\}$

$= \sup\{|x_i - z_i + z_i - y_i|\}$

$\leq \sup\{|x_i - z_i|\} + \{|z_i - y_i|\}$

$= \rho(x,z) + \rho(z,y)$

Hence it is a metric space $\square$

Example 7 Let C_0 be the set of all ~~sequence~~ sequence real numbers converging to zero and let ρ be a function defined on C_0 by $\rho(x,y) = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$. Then ρ is a metric on C_0 and so (C_0, ρ) is a metric space.

Varification

M1: $\rho(x,y) = \sup\{|x_i - y_i|\} = 0 \quad , \quad i \in \mathbb{N}$

$\Leftrightarrow \quad |x_i - y_i| = 0 \quad , \quad i \in \mathbb{N}$

$\Leftrightarrow \quad x_i - y_i = 0 \quad , \quad i \in \mathbb{N}$

$\Leftrightarrow \quad x_i = y_i \quad , \quad i \in \mathbb{N}$

$\Leftrightarrow \quad x = y$

M2: $\rho(x,y) = \sup\{|x_i - y_i|\} \geq 0$

being the supremum of non-negative numbers.

M3: $\rho(x,y) = \sup\{|x_i - y_i|\}$

$= \sup\{|y_i - x_i|\} = \rho(y,x)$

M4: $\rho(x,y) = \sup\{|x_i - y_i|\}$

$= \sup\{|x_i - z_i + z_i - y_i|\}$ for $i \in \mathbb{N}$

$\leq \sup\{|x_i - z_i|\} + \sup\{|z_i - y_i|\}$

$\leq \rho(x,z) + \rho(z,y)$

Hence ρ is a metric on C_0 and so (C_0, ρ) is a metric space. $\square$

DEFINITION: Let X be any non-empty set and let ρ be defined
EXM 9:

$$\rho(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

then ρ is a metric on X called a discrete metric and so (X,ρ) is a metric space called a discrete metric space.

Varification:

M1: $\rho(x,y) = 0 \Leftrightarrow x = y$, since $\rho(x,y) = 0$ then $x=y$ and if $x=y$ then $\rho(x,y) = 0$ by definition

M2: $\left.\begin{array}{l} \rho(x,y) = 1 \text{ if } x \neq y \\ \rho(x,y) = 0 \text{ if } x = y \end{array}\right\} \rho(x,y) \geq 0 \ \forall_{x,y \in X}$

M3: $\left.\begin{array}{l} \rho(x,y) = 1 = \rho(y,x) \text{ if } x \neq y \\ \rho(x,y) = 0 = \rho(y,x) \text{ if } x = y \end{array}\right\} \Rightarrow \rho(x,y) = \rho(y,x) \ \forall_{x,y \in X}$

M4: $\rho(x,y) = 1$ if $x \neq y$, $\rho(x,z) = 1$ if $x \neq z$ and $\rho(z,y) = 1$ if $z \neq y$ and so

$\rho(x,y) = 1 < 1+1 = \rho(x,z) + \rho(z,y)$ —— (1) also

$\rho(x,y) = 0$ if $x = y$, $\rho(x,z) = 0$ if $x = z$ and $\rho(z,y) = 0$ if $z = y$ and so

$\rho(x,y) = 0 = 0+0 = \rho(x,z) + \rho(z,y)$ —— (2)

and so from (1) and (2) we see that

$\rho(x,y) \leq \rho(x,z) + \rho(z,y) \quad \forall \ x,y,z \in X.$

29/1/2020

INEQUALITY: 1) **Hölders Inequality:**

Let x_i, y_i be any non-negative real numbers and $1 < p,\ q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\left(\sum_{i=1}^{\infty} |x_i y_i|\right) \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{1/q}$$

This is called the Hölder's Inequality for infinite sums. In addition, the schwart's Inequality is a special case of the Holder's inequality in which $p = q = 2$.

2) **Minkowski's Inequality:**

Let x_i, y_i be any two non-negative real numbers and $p \geq 1$ (i.e $1 \leq p \leq \infty$) then

$$\left(\sum_{i=1}^{\infty} (|x_i y_i|)^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}$$

proof:-

The case $p = 1$ is Immediate from the triangular Inequality for absolute value. So let $p > 1$ and define $q = \frac{p}{p-1}$ and let $D = \sum_{i=1}^{n} (|x_i| + |y_i|)^p$

observe that

$$(|x_i| + |y_i|)^p = (|x_i| + |y_i|) \cdot (|x_i| + |y_i|)^{p-1}$$

$$= |x_i| (|x_i| + |y_i|)^{p-1} + |y_i| (|x_i| + |y_i|)^{p-1}$$

so that

$$D = \sum_{i=1}^{n} |x_i| (|x_i| + |y_i|)^{p-1} + \sum_{i=1}^{n} |y_i| (|x_i| + |y_i|)^{p-1} \quad \text{---- (i)}$$

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$q = \frac{p}{p-1} \Rightarrow p = q(p-1)$

Applying the Holder's inequality seperately to the two pieces of the R.H.S of (a)

$$D \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} \left(|x_i| + |y_i|\right)^{(p-1)q}\right)^{1/q} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} \left(|x_i| + |y_i|\right)^{(p-1)q}\right)^{1/q}$$

$$D \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} \left(|x_i| + |y_i|\right)^p\right)^{1/q} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} \left(|x_i| + |y_i|\right)^p\right)^{1/q}$$

$$D \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p} \left[\left(\sum_{i=1}^{n} \left(|x_i| + |y_i|\right)^p\right)^{1/q}\right]$$

$$D \le \left[\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}\right] \cdot D^{1/q}$$

Divide both side by $D^{1/q}$

$\frac{D}{D^{1/q}} = D^{1-1/q}$

$$D^{1-\frac{1}{q}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}$$

$$D^{1/p} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}$$

Since

$$\sum_{i=1}^{n} \left(|x_i| + |y_i|\right)^p \le \sum_{i=1}^{n} \left(|x_i| + |y_i|\right)^p = D \quad \text{and} \quad 1 - \frac{1}{q} = \frac{1}{p}$$

we see that

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p} \quad \blacksquare$$

Example 8: Let $p \ge 1$ (i.e $1 \le p \le \infty$) be any real numbers and let l^p be the set of all sequences (x_i) of real numbers $\ni \sum_{i=1}^{\infty} |x_i|^p < \infty$. l^p is called p-summable set.
Now define ρ on l^p by

$$\rho(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} \text{ where } x=(x_i),\ y=(y_i) \in \ell^p.$$
then ρ is a metric ~~space~~ on ℓ^p and so (ℓ^p, ρ) is a metric space.

<u>Varification:-</u>

M1: $\rho(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} = 0$

$$\Longleftrightarrow \sum_{i=1}^{\infty} |x_i - y_i|^p = 0, \quad p \geq 1$$

$$\Longleftrightarrow |x_i - y_i|^p = 0$$

$$\Longleftrightarrow |x_i - y_i| = 0, \quad i = 1,2,3\ldots$$

$$\Longleftrightarrow x_i - y_i = 0 \quad i = 1,2,3 --$$

$$\Longleftrightarrow x_i = y_i \quad \forall\ i = 1,2,3 --$$

$$\Longleftrightarrow x = y$$

M2: $\rho(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} \geq 0$

being the positive root of non-negative number.

M3: $\rho(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p}$

$$= \left(\sum_{i=1}^{\infty} |y_i - x_i|^p\right)^{1/p} = \rho(y,x)$$

M4: $\rho(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p}$

$$= \left(\sum_{i=1}^{\infty} |x_i - z_i + z_i - y_i|^p\right)^{1/p}$$

$$\leq \left(\sum_{i=1}^{\infty} |x_i - z_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |z_i - y_i|^p\right)^{1/p}$$

By minKowski's inequality. $\therefore \rho(x,y) = \rho(x,z) + \rho(z,y)$ ■

REMARK: The sequence space l^p, $1 \le p \le \infty$ and the function space $C([a,b])$ are very important space in functional Analysis.

Exercises: 1) Consider the set $X = \mathbb{R}^n$ and let $\rho : X \times X \to \mathbb{R}$ be defined by $\rho(x,y) = \max_{1 \le i \le n} |x_i - y_i|$ where $x = (x_1, x_2, x_3 --- x_n)$ and $y = (y_1, y_2, y_3 --- y_n)$ are arbitrary elements of $\mathbb{R}^n$. Verify that (X, ρ) is a metric space.

2) Consider the set $X = \mathbb{R}^2$ and let $\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\rho(x,y) = \left(\sum_{i=1}^{2} (x_i - y_i)^2 \right)^{1/2}$. Verify that (X, ρ) is a metric space. This is generally referred to as the usual metric on $\mathbb{R}$ or Euclidean metric.

3) Verify that Schwartz's inequality i.e for any real number x_i, y_i, $i = 1, 2 --- n$ we have

$$\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |y_i|^2 \right)^{1/2}$$

4) Consider the set $C([a,b])$. Then verify that the mapping ρ defined by

$$\rho(f,g) = \left[\int_a^b (f(t) - g(t))^2 \right]^{1/2}$$ is a

metric and $([a,b], \rho)$ is a metric space.

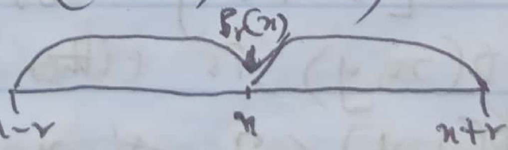
Open Ball's And Closed Balls

30/1/2020

Let (X, ρ) be any non-empty metric space and let $x \in (X, \rho)$ be any point. Now, for any positive real number r, the set $B_r(x)$, where $B_r(x) = \{y \in X : \rho(x,y) < r\}$ is called an open Ball in (X, ρ). And the set $\bar{B}_r(x)$ where $\bar{B}_r(x) = \{y \in X, \rho(x,y) \leq r\}$ is called a closed ball in (X, ρ).

Example 1: Let $(x, y) = (\mathbb{R}, |\cdot|)$ and let $r > 0$ (so $0 < r \in \mathbb{R}$) then by the above definition

$$
\begin{aligned}
B_r(x) &= \{y \in X : \rho(x,y) < r\} \\
&= \{y \in X : |x - y| < r\} \\
&= \{y \in \mathbb{R} : -r < y - x < r\} \\
&= \{y \in \mathbb{R} : x - r < y < x + r\} \\
&= (x - r, x + r)
\end{aligned}
$$



this shows that $B_r(x)$ in $\mathbb{R}$ is always $B_r(x) = (x - r, x + r)$ and $\bar{B}_r(x) = [x - r, x + r]$.

Example 2: Let (X, ρ) be a discrete metric. If

$$\rho(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$ for arbitrary $x, y \in \mathbb{R}$.

Describe the open balls:

i) $B_{1/2}(1)$ ii) $B_2(1)$ iii) $B_1(5)$ iv) $B_{3/2}(4)$

soln

i) By definition

$$B_{1/2}(1) = \{y \in \mathbb{R} : \rho(1, y) < 1/2\}$$

But $\rho(x,y)$ has only two points.

viz: 0 or 1 and $\frac{1}{2} < 1 \Rightarrow \rho(1,y) = 0$.

$$\therefore B_{1/2}(1) = \{y \in \mathbb{R} : \rho(1,y) = 0\}.$$

By condition M1, we see that

$$\rho(1,y) = 0$$

$$\Rightarrow y = 1$$

$$B_{1/2}(1) = \{y \in \mathbb{R} : \rho(1,y) = 0\} = \{y \in \mathbb{R} : y = 1\} = \{1\}$$

Hence $B_{1/2}(1) = \{1\}$, the singleton set $\{1\}$.

ii) By definition

$$B_2(1) = \{y \in \mathbb{R} : \rho(1,y) < 2\}$$ but

$\rho(x,y)$ is either 0 or 1 and in any case

$$\rho(1,y) < 2 \quad \forall\, x, y \in \mathbb{R}$$

Thus $B_2(1) = \{y \in \mathbb{R} : \rho(1,y) < 2\} = \{y \in \mathbb{R}\} = \mathbb{R}$

Thus $B_2(1) = \mathbb{R}$ (the whole space $\mathbb{R}$).

Open Sets

Let A be any subset of a metric space (X, ρ).

Then A is said to be open (or is an open set in (X,ρ))

if $\forall\, a \in A\ \exists\, r > 0 \ni B_r(a) \subseteq A$.

✓ The Intersection $X \cap Y$ is formed by all points which are elements of both X and Y.
✓ The union $X \cup Y$ consists of all points which are elements of either X or Y, including elts of both.
✓ The complement of a set X consist of all points which are not in X. It is denoted by $\sim X$.

✓ A set is a collection of identifiable objects, it's elements.
✓ A set can be referred to as a space & an element as a point.
✓ Two sets are equal iff they have the same elements $X = Y$
✓ X is a subset of Y if every element of X is also an element of Y, $X \subset Y$ or $Y \supset X$.

Lemma 1.0: Let (X,ρ) be any metric space. Then each open ball is an open set in X.

proof:

Let $B_r(x)$ be any arbitrary open ball in (X,ρ). We want to prove that $B_r(x)$ is an open set. So, we must show that for any point say $y \in B_r(x)$ we can find some $r_1 > 0 \ni B_{r_1}(y) \subseteq B_r(x)$.

So, let y be arbitrary chosen in $B_r(x)$ define $r_1 = r - \rho(x,y)$. Since $y \in B_r(x)$ then $r - \rho(x,y) > 0$ $\Rightarrow r_1 > 0$.

We assert that $B_{r_1}(y) \subseteq B_r(x)$.

To prove the assertion, it suffices to show that every element in $B_{r_1}(y)$ is an element of $B_r(x)$. So, let $y_1 \in B_{r_1}(x)$ be arbitrary. Then to show that $y_1 \in B_r(x)$ we must show that $\rho(x,y_1) < r$.

Now, $y \in B_{r_1}(y) \Rightarrow \rho(y,y_1) < r_1$.

Using the triangular inequality

$$\rho(x,y_1) \leq \rho(x,y) + \rho(y,y_1)$$
$$< r_1 + \rho(x,y)$$
$$= (r - \rho(x,y)) + \rho(x,y)$$
$$= r.$$

i.e. $\rho(x,y) < r$ and so $y_1 \in B_r(x)$

Thus, $y \in B_{r_1}(y) \Rightarrow y \in B_r(x) \Rightarrow B_{r_1}(y) \subseteq B_r(x)$.

Since the point $y \in B_{r_1}(x)$ is arbitrary there result follows. □

6/2/20

Lemma 1.1 Let (X, ρ) be any metric space. Then
i) The arbitrary union of open set in X is open.
ii) The finite intersection of open sets in X is open.

proof: (i)
Let $\{B_\alpha : \alpha \in I\}$ (where I is an arbitrary Index set) be a collection of open sets in (X, ρ).
Let $B = \cup_\alpha B_\alpha$ the arbitrary union of open set in (X, ρ). Let $x \in B = \cup_\alpha B_\alpha$ be arbitrary. But $x \in B(\cup_\alpha B_\alpha) \Rightarrow x \in B_{\alpha_0}$ for some $\alpha_0 \in I$.
Since B_{α_0} is open then $\exists\ r > 0 \ni$
$$B_r(x) \subseteq B_{\alpha_0} \subseteq \cup_\alpha B_\alpha \quad \text{(by definition)}.$$
$\Rightarrow \forall x \in \cup_\alpha B_\alpha \ \exists\ r > 0 \ni$
$$B_r(x) \subseteq \cup_\alpha B_\alpha$$
Hence $B = \cup_\alpha B_\alpha$ is open set in X. ■

proof (ii):
Let $\{B_i : i = 1, 2, \ldots n\}$ be a finite collection of open sets in (X, ρ). Now let $x \in \bigcap_{i=1}^{n} B_i$ be arbitrary then $x \in B_i$ for each i. so $\exists\ r > 0 \ni$
$$x \in B_r(x) \subseteq B_i \text{ for } i = 1, 2, \ldots n$$
i.e $B_{r_1}(x) \subseteq B_1$, $B_{r_2}(x) \subseteq B_2$ ...... $B_{r_n}(x) \subseteq B_n$
Let $r = \min\{r_1, r_2, \ldots r_n\}$. Then
$$B_r(x) \subseteq B_{r_1}(x) \subseteq B_{r_2}(x) \subseteq \ldots \subseteq B_{r_n}(x).$$
$\Rightarrow B_r(x) \subseteq B_{r_i}(x) \subseteq \bigcap_{i=1}^{n} B_{r_i}(x)$.
Hence $\bigcap_{i=1}^{n} B_i$ is open in X. ■

Interior point of a set X is the largest open set contained in X, it is also the union of all open sets ⊆ X, it is also described as the set of all points of which X is a neighbourhood.

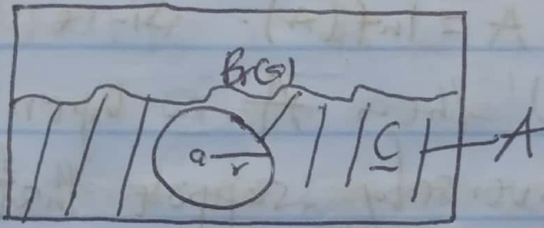
DEFINITION: INTERIOR POINT

Let A be any subset of an arbitrary (non empty) metric space (X, ρ) then a point $a \in A$ is said to be an interior point of A if $\exists\ r > 0 \ni$

$$B_r(a) \subseteq A.$$

The set of all interior points of A is called the interior of A and is denoted by $\text{Int}(A)$ or A° i.e

$$\text{Int}(A) = \{a \in A : B_r(a) \subseteq A \text{ for some } r > 0\}.$$



LEMMA 1.2 Let A be any subset of an arbitrary metric space (X, ρ) then

i. $\text{Int}(A)$ is open set

ii. $\text{Int}(A)$ is the largest open set contained in A.

iii. A is open iff $\text{Int}(A) = A$.

iv. If $A \subseteq B$ then $\text{Int}(A) \subseteq \text{Int}(B)$.

proof (i):

Let $x \in \text{Int}(A)$ then x is an interior point of A then by definition $\exists\ r > 0 \ni x \in B_r(x) \subseteq A$ i.e every point of $B_r(x)$ is an interior point of A and so $B_r(x) \subseteq \text{Int}(A)$. ~~Thus~~,

Thus, $\text{Int}(A)$ is an open set. ∎

proof (ii):

Let B be any open set contained in A and let $x \in B$. Since B is open then by definition $\exists\, r>0 \ni B_r(x) \subseteq B$. Since every open ball is an open set then $x \in B_r(x) \subseteq \text{Int}(A)$. But $x \in B$ is arbitrary and so $B \subseteq \text{Int}(A)$.

$\Rightarrow \text{Int}(A)$ is the largest open set contained in A. $\square$

proof (iii):

Let $A = \text{Int}(A)$. Since $\text{Int}(A)$ is an open set by (i) then A is open.

Conversely, suppose that A is open then by definition $\forall\, a \in A\ \exists\, r>0 \ni B_r(a) \subseteq A$. Since every open ball is an open set, then

$a \in B_r(a) \subseteq \text{Int}(A)$.

Since $a \in A$ is arbitrary, then $A \subseteq \text{Int}(A)$ but $\text{Int}(A) \subseteq A$ always and so $A = \text{Int}(A)$.

proof (iv):

Let $A \subseteq B$ and let $x \in \text{Int}(A)$. Then x is an interior point of A. i.e $\exists\, r>0 \ni x \in B_r(x) \subseteq A$. Since $A \subseteq B$ then $x \in B_r(x) \subseteq A \subseteq B$. But this implies that $x \in \text{Int}(B)$ and so $\text{Int}(A) \subseteq \text{Int}(B)$

Exercise: Show that $\text{Int}(A) \cup \text{Int}(B) \subseteq \text{Int}(A \cup B)$ for any A, B in (X, ρ).

Closed sets

12/2/20

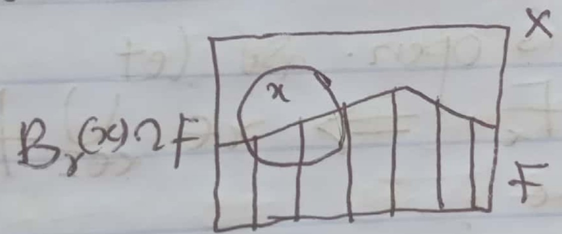
Before we define closed sets we give some other definitions.

DEFINITION: (Cluster point or limit point)

Let F be any subset of an arbitrary (non-empty) metric space (X, ρ). Then a point $x \in X$ (not necessary in F) is said to be a cluster point (or limit point) of F, if $\exists$ $r>0$ $\ni$

$$B_r(x) \cap F \neq \phi$$

The set of all cluster points of F is denoted by $\bar{F}$ and is called the closure of F.



DEFINITION: A subset F of a metric space (X, ρ) is said to be closed if it contains all its limit points.

REMARK:

~~A~~ subset F of a metric space is said to be closed if its complement F^c is open.

NOTE
① $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ ③ $\sim(X \cup Y) = \sim X \cap \sim Y$
② $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ ④ $\sim(X \cap Y) = \sim X \cup \sim Y$.

Lemma 1.3: Let (X, ρ) be any metric space. Then
i. The arbitrary intersection of closed sets in X is closed
ii. The finite Union of closed sets in X is closed.

proof:
Let $\{F_\alpha : \alpha \in I\}$ be a collection of closed sets in (X, ρ). Then we must to show that $\bigcap_{\alpha\in I} F_\alpha$ is closed. To see this, Consider $\bigcap_{\alpha\in I} F_\alpha$.
Since F_α is closed then each F_α^c is opened. So,

$$\left(\bigcap_{\alpha\in I} F_\alpha\right)^c = \bigcup_{\alpha\in I} F_\alpha^c$$ by Demorgans law.

But we have seen that arbitrary Union of open sets is open. So let

$$x \in \bigcap_{\alpha\in I} F_\alpha \Rightarrow x \in \bigcup_{\alpha\in I} F_\alpha^c$$

Since $\bigcup_{\alpha\in I} F_\alpha^c$ is open then $\exists\ r > 0 \ni$

$$B_r(x) \subseteq \bigcup_{\alpha\in I} F_\alpha^c \Rightarrow B_r(x) \subseteq \bigcap_{\alpha\in I} F$$

Hence $\bigcap_{\alpha\in I} F_\alpha$ is closed. $\square$

(ii) proof as an (exercise)

Lemma 1.4 1) $\bar{F}$ is closed.

proof:
To see that $\bar{F}$ is closed, we need to show that $\bar{F}^c$ is open. To see this Let $\bar{F}^c \neq \phi$

and Let $x \in \bar{F}^c$ then $x \notin \bar{F}$ and so, $\exists\ r>0 \ni$

$$B_r(x) \cap F = \phi$$

Now let $y_1 \in B_r(x)$ be arbitrary $\rho(y_1, x) < r$ and so Let $r_1 = r - \rho(y_1, x) > 0$ then

$$B_{r_1}(y_1) \subseteq B_r(x) \cdot \text{[show This or do this]}$$

then $B_{r_1}(y_1) \cap F = \phi$ then $y_1 \notin \bar{F}$ and so $x \in \bar{F}^c$ or $B_{r_1}(x) \subseteq F$. This shows that F^c is open and so $\bar{F}$ is closed. □

2) if A, B are in $(X, \rho) \ni A \subseteq B$ then $\bar{A} \subseteq \bar{B}$.

<u>proof</u>

Let $x \in \bar{A}$ then by definition $\exists\ r>0 \ni$

$$B_r(x) \cap A \neq \phi.$$

since $A \subseteq B$ ~~then~~ $B_r(x) \cap B \neq \phi$ and so $x \in \bar{B}$. since $x \in \bar{A}$ is arbitrary then $\bar{A} \subseteq \bar{B}$.

DEFINITION: Let F be any subset of an arbitrary metric space (X, ρ). Then a point $x \in X$ (not necessary in F) is said to be an <u>accumulation point</u>, If $\exists\ r>0 \ni$

$$B_r(x) \cap (F \setminus \{x\}) \neq \phi$$

3) F is closed iff $F = \bar{F}$.

<u>proof</u>:

If $F = \bar{F}$ then F is closed since $\bar{F}$ is closed. Conversely, suppose F is closed then F^c is open.

Since $F \subseteq \bar{F}$ we only need to show that $\bar{F} \subseteq F$. To see this let $x \in \bar{F}$ then $\exists\, r>0 \ni B_r(x) \cap F \neq \emptyset \Rightarrow x \in F$.

Hence $\bar{F} \subseteq F$. Thus $F = \bar{F}$.

4) Show that $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

proof:

Let $x \in \overline{A \cap B} \Rightarrow x$ is a closure of $A \cap B$

i.e. $\forall\, x \in X$, $\exists\, r>0 \ni B_r(x) \cap (A \cap B) \neq \emptyset$

i.e. $B_r(x) \cap A \cap B_r(x) \cap B \neq \emptyset$.

$\Rightarrow$ $B_r(x) \cap A$ and $B_r(x) \cap B \neq \emptyset$

$\Rightarrow$ x is a closure of A and x is a closure of B.

$\Rightarrow$ $x \in \bar{A}$ and $x \in \bar{B}$

$\Rightarrow$ $x \in \bar{A} \cap \bar{B}$

Hence $x \in \overline{A \cap B} \Rightarrow x \in \bar{A} \cap \bar{B}$

Thus $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$. $\square$

SEQUENCES IN METRIC SPACES

DEFINITION: Let (x_n) be any sequence in a metric space (X, ρ) then (x_n) is said to converge to a point x in X, if $\forall\, \varepsilon>0\ \exists\, n_0 \in \mathbb{N} \ni \rho(x_n, x) < \varepsilon\ \forall\, n \geq n_0$.

We write this as $\lim x_n = 0$ or $\lim\limits_{n\to\infty} \rho(x_n, x) = 0$ or $x_n \underset{n\to\infty}{\to} x$ in (X, ρ) or $x_n \in B_\varepsilon(x)$, $\forall\, n \geq n_0$.

DEFINITION: A sequence (x_n) in a metric space (X, ρ) is said to be Cauchy if $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ni \rho(x_n, x_m) < \varepsilon \ \forall n, m \geq n_0$.

i.e $\lim_{n \to \infty} \rho(x_n, x_m) = 0$.

REMARK

Note that in general metric space every convergent sequence is Cauchy. But every Cauchy sequence need not converge to the point of the space.

examples seq $(1/n)$ in $((0,1), |\cdot|)$ converges to 0 but $0 \notin (0,1)$.

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LEMMA 1.5: Every convergent sequence (x_n) in a metric space (X, ρ) is Cauchy.

proof:

Suppose the sequence (x_n) converge to x. Then by definition $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ni \rho(x_n, x) < \varepsilon/2 \ \forall n \geq n_0$ —— (1)

Now for $m \geq n_0$, we have by (1)

$\rho(x_m, x) < \varepsilon/2 \quad \forall m \geq n_0$ —— (2)

from (1) and (2) we have

$$\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x_m, x) < \varepsilon$$

$\forall n, m \geq n_0$

Hence (x_n) is Cauchy. □

DEFINITION: A metric space (X, ρ) is said to be <u>complete</u> if every Cauchy sequence in X converges to a point of X.

LEMMA 1.6: Let F be any subset of a metric space (X, ρ). Then $x \in \bar{F}$ iff $\exists (x_n) \subset F \ni x_n \xrightarrow[n \to \infty]{} x$.

<u>proof:</u>

Let $x \in \bar{F}$ then by definition $\exists\ r > 0 \ni$

$$B_r(x) \cap F \neq \phi.$$

Now in particular for $r = \frac{1}{n}$ we see that $B_{1/n}(x) \cap F \neq \phi$

i.e $\forall\ n \in \mathbb{N}\ \exists\ (x_n) \subseteq F \ni x_n \in B_{1/n}(x)$

i.e $\rho(x_n, x) < 1/n \xrightarrow[n \to \infty]{} 0 \implies x_n \xrightarrow[n \to \infty]{} x$.

Conversely, suppose that $\exists\ (x_n) \subseteq F \ni x_n \xrightarrow[n \to \infty]{} x$

then by definition $\forall\ \varepsilon > 0\ \exists\ n_0 \in \mathbb{N} \ni \rho(x_n, x) < \varepsilon$ $\forall\ n \geq n_0$. so for $n = n_0$ we get $\rho(x_{n_0}, x) < \varepsilon$ and in particular for $\varepsilon = r$, we get

$$\rho(x_{n_0}, x) < r$$

$x_{n_0} \in B_r(x)$ and so $B_r(x) \cap F \neq \phi$.

since $x_{n_0} \in B_r(x)$. Thus $x \in \bar{F}$. $\square$

DEFINITION: Let A be any subset of metric space (X, ρ) then A is a metric subspace of (X, ρ) if $\rho_A = \rho \restriction A$. i.e (A, ρ_A) is a metric space and is a subspace of (X, ρ).

$$\rho \restriction A = \rho_A : A \times A \longrightarrow \mathbb{R}$$

Lemma 1.7: A subset F of a complete metric space (X, ρ) is closed iff F is complete.

proof:

Suppose that $F \subseteq (X, \rho)$ is complete i.e every Cauchy sequence in F converges to a point of F. We want to show that F is closed.

We know that $F \subseteq \bar{F}$. So to show that F is closed we need to show that $\bar{F} \subseteq F$.

To see this, let $x \in \bar{F}$ then $\exists (x_n) \subseteq F \ni x_n \xrightarrow[n\to\infty]{} x$. Since $F \subseteq (X, \rho)$ then (x_n) is in (X, ρ). Since (x_n) converges, then it is Cauchy since F is complete. and (x_n) is Cauchy and $x_n \xrightarrow[n\to\infty]{} x$ then $x \in F$. Hence $x \in \bar{F} \Rightarrow x \in F$.

Thus $\bar{F} \subseteq F$ i.e F is closed.

Conversely, suppose that F is closed then we want to show that F is complete.

To see this, let (x_n) be arbitrary sequence in F. Since $F \subseteq X$ then (x_n) is Cauchy sequence in X. Since X is complete and $x_n \to x$ then $x \in \bar{F}$. Since F is closed then $\bar{F} = F$ i.e $x \in \bar{F} = F$. So (x_n) is a Cauchy sequence in F and $x_n \xrightarrow[n\to\infty]{} x \in F$.

Thus, F is complete $\square$

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DEFINITION: Let (X,ρ) be an arbitrary metric space. A mapping $T: X \to X$ is called a <u>contraction map</u> if $\forall\, x,y \in X$, $\exists$ a constant $K \in [0,1)$ $\ni$

$$\rho(T(x), T(y)) \le K\rho(x,y)$$

LEMMA 1.8: Every contraction mapping is continuous

<u>proof:</u>

Let $T: (X,\rho) \to (X,\rho)$ be a contraction mapping, then $\exists$ a ~~con~~ constant $K<1$ $\ni$

$$\rho(T(x), T(y)) \le K\rho(x,y) \quad \forall\, x,y \in X.$$

Let $\varepsilon > 0$ be given, we want to find $\delta > 0$ $\ni$ $\rho(x,y) < \delta \Rightarrow \rho(T(x), T(y)) < \varepsilon$. Choose $\delta = \varepsilon$ then $\rho(T(x), T(y)) \le K\rho(x,y) < K\delta = K\varepsilon < \varepsilon$. since $K<1$

Thus, $\rho(x,y) < \delta \Rightarrow \rho(T(x), T(y)) < \varepsilon$.

Hence T is continuous. $\square$

DEFINITION: Let (X,ρ) be a metric space and let $T: X \to X$ be any mapping of X into itself. A point x^* is called <u>fixed point</u> of T if

$$T(x^*) = x^*$$

LEMMA 1.9: (Contraction mapping principle)

Every contraction mapping on a complete metric space has a unique fixed point.

<u>proof:</u>

Let (X, ρ) be a Complete metric space and let $T: X \longrightarrow X$ be a Contraction mapping.

<u>Step 1:</u> Generate a seq $\{x_n\}$ in X. Let $x_0 \in X$ be arbitrary point then define

$$x_1 = T(x_0)$$
$$x_2 = T(x_1) = T(T(x_0)) = T^2(x_0)$$
$$x_3 = T(x_2) = T^3(x_0)$$
$$\vdots$$
$$x_n = T(x_{n-1}) = T^n(x_0)$$

Clearly, the sequence $\{x_n\} \subseteq X$.

<u>Step 2:</u> Show that $\{x_n\}$ is a Cauchy sequence.

Let $x_n = T(x_{n-1})$ and $x_{n+1} = T(x_n)$ then

$$\rho(x_n, x_{n+1}) = \rho(T(x_{n-1}), T(x_n))$$
$$\leq K\rho(x_{n-1}, x_n)$$
$$= K\rho(T(x_{n-2}), T(x_{n-1}))$$
$$\leq K^2\rho(x_{n-2}, x_{n-1})$$
$$\vdots$$
$$= K^n\rho(x_0, x_1)$$ and

$$\rho(x_n, x_{n+1}) \leq K^n\rho(x_0, x_1) \quad \forall n \text{ ---- (i)}$$

We can now show that $\{x_n\}$ is Cauchy. Let $m > n$ then

$$\rho(x_n, x_m) \leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{m-1}, x_m)$$
$$\leq K^n\rho(x_0, x_1) + K^{n+1}\rho(x_0, x_1) + \cdots + K^{m-1}\rho(x_0, x_1)$$
$$= K^n\rho(x_0, x_1)[1 + K + K^2 + \cdots K^{m-1-n}]$$
$$\leq K^n\rho(x_0, x_1)[1 + K + K^2 + \cdots + K^{m-1} + K^m + \cdots]$$

but $1+K+K^2+\dots+K^{m-1}+K^m+\dots$ is G.P with $K<1$. It sum to infinity $\frac{1}{1-K}$. and so

$$\rho(x_n, x_m) \le K^n \rho(x_0, x_1)\left(\frac{1}{1-K}\right) \longrightarrow 0 \text{ as}$$

$n \longrightarrow \infty$ since $K<1$.

$\Rightarrow (x_n)$ is Cauchy in X and since X is complete, (x_n) converges to a point in X.

STEP 3: Since x^* is a fixed point of T.

Let $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$ ---- (2)

Since T is contraction and hence is cts, it follows ~~from~~ (2) that

$$T(x_n) \longrightarrow T(x^*) \text{ as } n \longrightarrow \infty.$$

But $x_{n+1} = T(x_n)$ and so

$$x_{n+1} = T(x_n) \longrightarrow T(x^*) \text{ ---- } (3)$$

But the limit are unique in a metric space.

so from (2) and (3) we obtain

$$T(x^*) = x^* \text{ ------- } (4)$$

Hence, T has a fixed point in X.

STEP 4: The fixed point is unique.

Suppose $x^* \neq y^*$ and $T(y^*) = y^*$ ---- (5)

from (4) and (5)

$$\rho(x^*, y^*) = \rho(T(x^*), T(y^*))$$

$$\rho(x^*, y^*) \le K\rho(x^*, y^*)$$

$$0 \le K\rho(x^*, y^*) - \rho(x^*, y^*)$$

$$(K-1)\rho(x^*, y^*) \ge 0 \quad \text{and} \quad \rho(x^*, y^*) \ne 0$$

$$\Rightarrow \quad K - 1 \ge 0 \quad \Rightarrow K \ge 1 \quad (\text{Contradiction.})$$

Hence $x^* = y^*$ (uniqueness).

REMARK: The Contraction mapping is called non-expansive if $K=1$ i.e. T is non expansive if $\forall\, x, y \in X$

$$\rho(T(x), T(y)) \le \rho(x, y)$$

Clearly, every Contraction mapping is non-expansive but the Converse is not true. $\square$

EXERCISE: Let $X = [4, \infty)$ with usual metric function $\mathbb{R}$ and let $T: X \longrightarrow \mathbb{R}$ be defined by

$$T(x) = \frac{1}{2}\left(x + \frac{16}{x}\right) \text{ for } x \in [4, \infty).$$

prove: i) X is a Complete metric space

ii) T is a Contraction mapping on X

iii) What is the unique fixed point of T.

2) Let $X = [0, \infty)$ be endowed with the usual metric fxn $\mathbb{R}$, and let $T: X \to \mathbb{R}$ be defined by

$$T(x) = x + \frac{1}{x} + \frac{1}{x^3}$$

prove: (i) X is a Complete metric space

ii) $|T(x) - T(y)| \le |x - y| \quad \forall\, x, y \in X$.

iii) T has no fixed point.

Why does (iii) above not Contradict the Contraction mapping principle.

NESTED INTERVAL: Let $I = [a_n, b_n]$ be closed bounded interval in R $\ni$ $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$

i.e The interval $[a_n, b_n]$ is nested for each $n \in \mathbb{N}$.

i.e $a_n \le a_{n+1} \le a_{n+2} \le \cdots$ $b_n \ge b_{n+1} \ge b_{n+2} \ge \cdots$

$[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ $a_n \le a_{n+1} \le b_{n+1} \le b_n \; \forall n$

and $\lim_{n\to\infty}(b_n - a_n) = 0$ then $\bigcap_{n=1}^{\infty} I_n$ contains exactly one p

proof:

Since I_n is nested then (a_n) is an increasing monotonic sequence that is bounded above.

So, let $\sup a_n = x$ then $a_n \le x$ $\lim_{n\to\infty} a_n = x$.

Also (b_n) is decreasing monotonic sequence that is bounded below. so let $\inf b_n = y$ then $y \le b_n$

and $\lim_{n\to\infty} b_n = y$.

Since $\lim_{n\to\infty}(b_n - a_n) = 0$ then

$y - x = 0 \Rightarrow x = y$. Since

$a_n \le x \le b_n \quad \forall n$ then

$$\bigcap_{s=1}^{\infty} [a_n, b_n] = \{x\}.$$

To show the uniqueness, suppose that there is another point say $z \in [a_n, b_n] \; \forall n$

i.e $\{z\} = \bigcap_{s=1}^{\infty} [a_n, b_n]$

$a_n \le x \le b_n$

$0 \le x - a_n \le b_n - a_n$

$a_n \le z \le b_n \Rightarrow -b_n + a_n \le z - b_n \le 0$

Now, $-b_n + a_n \le z - x \le b_n - a_n$

$\Rightarrow |z-x| \le b_n - a_n$ since $\lim_{n\to\infty}(b_n - a_n) = 0$

$\Rightarrow |z-x| = 0 \Rightarrow x = z$

Hence, x is unique. $\square$

Bounded Convergence Theorem 20/2/20

DEFINITION: Let F be any subset of a metric space (X, ρ). Then $d(F)$ or $\text{diam}(F)$ is defined as

$d(F) = \sup\{\rho(x,y) : x, y \in F\}$.

Cantor's Intersection (or Nested) Theorem

Lemma 2.0: Let (F_n) be a sequence of closed subsets such that $\ni F_{n+1} \subseteq F_n \ \forall n$ and that $\text{diam}(F) \xrightarrow[n\to\infty]{} 0$ then $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point or ($\bigcap_{n=1}^{\infty} F_n \neq \phi$ or $\bigcap_{n=1} F_n$ is a singleton set).

Proof:

Since $F_{n+1} \subseteq F_n \ \forall n$ then $F_n \neq \phi$. Now let $x_1 \in F_1$, $x_2 \in F_2$, $x_3 \in F_3$ --- $x_n \in F_n$ --- Since $F_{n+1} \subseteq F_n \ \forall n$ then x_1, x_2, x_3 --- x_n, x_{n+1} all lies in F_n. Since $\text{diam}(F) \xrightarrow[n\to\infty]{} 0$ then by definition $\exists \ n_0 \in \mathbb{N} \ \ni$

$d(F) < \varepsilon \ \forall \ n \ge n_0$.

Now let $m > n \ge n_0$ then $F_m \subset F_n \subseteq F_{n_0}$

so $\rho(x_n, x_m) \le \sup\{\rho(x_n, x_m) : x_n, x_m \in F_{n_0}\} = d(F_{n_0}) < \varepsilon$

$\Rightarrow (x_n)$ is Cauchy in F_{n_0} and so (x_n) is Cauchy in X, $F_{n_0} \subseteq X$.

Since X is complete metric space, and (x_n) is Cauchy then $\exists\ x \in X \ni x_n \xrightarrow[n\to\infty]{} x$

since $(x_n) \subset F_{n_0}$ and F_{n_0} is closed and $x_n \xrightarrow[n\to\infty]{} x$

$\Rightarrow x \in \bar{F}_{n_0} = F_{n_0}$

$\Rightarrow x \in F_n\ \forall\ n$ and so $\bigcap_{n=1}^{\infty} F_n = \{x\}$.

<u>Uniqueness</u>: suppose there is another point $y \in F_n$, $\forall\ n$. Then $\rho(x, y) \leq d(F) \xrightarrow[n\to\infty]{} 0$

$\Rightarrow 0 \leq \rho(x, y) \leq 0$

$\Rightarrow \rho(x, y) = 0$ Hence $x = y$. $\square$

LEMMA 2.1: If a sequence (x_n) in a metric space (X, ρ) converges to a point then ~~every~~ every subsequence (x_{n_k}) of (x_n) converges to the ~~same~~ point ~~on~~ (x_n).

<u>**proof:**</u>

Let $x_n \xrightarrow[n\to\infty]{} x$ then by definition $\forall\ \varepsilon > 0\ \exists\ n_0 \in \mathbb{N} \ni \rho(x_n, x) < \varepsilon\ \forall\ n \geq n_0$ ——— $(*)$

Now let $k \geq n_0$ then $n_k \geq k \geq n_0$ and so by $(*)$

$\rho(x_{n_k}, x) < \varepsilon\ \forall\ n_k \geq n_0$

$\Rightarrow x_{n_k} \xrightarrow[n\to\infty]{} x$. $\square$

LEMMA 2.2: Let (x_n) be a Cauchy sequence in a metric space (X, ρ). If a subsequence (x_{n_k}) of (x_n) converges to a point then (x_n) also converges to the same point on (x_{n_k}).

proof:

Let (x_{n_k}) converges to x then by definition $\forall\ \varepsilon>0$ $\exists\ n_0 \in \mathbb{N} \ni \rho(x_{n_k}, x) < \varepsilon\ \forall\ k \geq n_0$. Since (x_n) is Cauchy sequence then by definition $\forall\ \varepsilon>0$ $\exists$ $n_0 \in \mathbb{N} \ni \rho(x_n, x_m) < \varepsilon\ \forall\ n, m \geq n_0$ ---- (*)

Now $k \geq n_0$ then $n_k \geq k \geq n_0$ and also putting $m = n_k$ in (*) we got $\rho(x_n, x_{n_k}) < \varepsilon/2\ \forall\ n \geq n_0$.

Hence $\forall\ n \geq n_0$ we have

$$\rho(x_n, x) \leq \rho(x_n, x_{n_k}) + \rho(x_{n_k}, x)$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$\Rightarrow \rho(x_n, x) < \varepsilon\ \ \forall\ n \geq n_0.$$

$$\Rightarrow x_n \xrightarrow[n\to\infty]{} x.$$ $\square$

THEOREM: BOLZANO - WEIERSTRASS THEOREM

Every bounded infinite subset of a real number has atleast one limit point.

proof:

Let A be any bounded infinite subset of $\mathbb{R}$. Then $\exists$ a constant $m, M \ni m \leq a \leq M\ \forall\ a \in A$.

Now consider the interval $[m, \frac{1}{2}(m+M)]$ or $[\frac{1}{2}(m+M), M]$. In these intervals, one of them has infinitely many points of A.

Let us take $[m, \frac{1}{2}(m+M)]$ and call it $[a_1, b_1]$ and consider $[a_1, \frac{1}{2}(a_1+b_1)]$ or $[\frac{1}{2}(a_1+b_1), b_1]$. Then one of these intervals has infinitely many points of A. Let us take $[a_1, \frac{1}{2}(a_1+b_1)]$ and call it $[a_2, b_2]$ then $[a_2, \frac{1}{2}(a_2+b_2)]$ contains infinitely many points of A and call it $[a_3, b_3]$. Continuing in this way, we get sequence $([a_n, b_n]) = I_n$ of closed bounded intervals $\ni I_1 \supseteq I_2 \supseteq I_3 \dots \supseteq I_n$ and that $b_n - a_n = \dfrac{M-n}{2^n} \xrightarrow[n\to\infty]{} 0$.

Then by nested interval theorem $\bigcap_{n=1}^{\infty} [a_n, b_n]$ contains exactly one point say x_0 i.e. $\bigcap_{n=1}^{\infty} [a_n, b_n] = \{x_0\}$.
Next we show that x_0 is a limit point of A.
Since $\lim_{n\to\infty} b_n = 0$ then let $\varepsilon > 0$ be given and choose $n \in \mathbb{N} \ni b_n - a_n < \varepsilon$ and consider the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ since $x_0 \in [a_n, b_n]$ $\forall n$ then $[a_n, b_n] \subset (x_0 - \varepsilon, x_0 + \varepsilon)$. And since $[a_n, b_n] \cap A \neq \phi$ then $A \cap (x_0 - \varepsilon, x_0 + \varepsilon) \neq \phi$
i.e. $A \cap B_\varepsilon(x_0) \neq \phi$
$\Rightarrow x_0$ is a limit point of A $\square$.

Completeness

26/2/20

Example 1 $(B([a,b]), \rho)$ is a complete metric space where $\rho(f,g) = \sup\{|f(t)-g(t)| : t \in [a,b]\}$

proof:

Let $\{f_n\}$ be a Cauchy sequence in $B([a,b])$. Then by definition $\forall \varepsilon > 0 \ \exists \ n_0 \in \mathbb{N} \ni$

$\rho(f_n, f_m) = \sup\{|f_n(t) - f_m(t)| : t \in [a,b]\} < \varepsilon \ \forall \ n,m \geq n_0$

$\Rightarrow |f_n(t) - f_m(t)| < \varepsilon \ \forall \ n,m \geq n_0$ —— (1)

and $\forall$ fixed $t \in [a,b]$.

$\Rightarrow \{f_n(t)\}$ is a Cauchy in $\mathbb{R}$ and so it converges in $\mathbb{R}$ to a function say $f(t)$ in $\mathbb{R}$.

Now letting $m \to \infty$ in (1) we have

$|f_n(t) - f(t)| < \varepsilon \ \forall \ n \geq n_0$.

$\Rightarrow \sup\{|f_n(t) - f(t)| : t \in [a,b]\} < \varepsilon \ \forall \ n \geq n_0$.

$\Rightarrow \rho(f_n, f) < \varepsilon \ \forall \ n \geq n_0$ and hence $\{f_n\}$ converges to f.

It remains to show that f is bounded. Since f_{n_0} is bounded then $\exists \ c > 0 \ni |f_{n_0}(t)| \leq c \ \forall \ t \in [a,b]$

$\therefore |f(t)| \leq |f(t) - f_{n_0}(t)| + |f_{n_0}(t)| < \varepsilon + c \doteq M$

when $M < \varepsilon + c$

$\Rightarrow |f(t)| \leq M \ \forall \ t \in [a,b]$.

$\Rightarrow f \in B([a,b])$ and hence $(B([a,b]), \rho)$ is a complete metric space. $\square$

NOTE: for $C([a,b])$ the proof is same as above but we show that it is continuous instead bdd. i.e. f is cont at $x_0 \in X$, if $\forall\ \varepsilon > 0\ \exists\ \delta > 0 \ni$ $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$.

Example 2: (C, ρ) is a complete metric space where ρ is defined as $\rho(x,y) = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$.

proof:
Let (x_n) be any Cauchy sequence in C, then by definition $\forall\ \varepsilon > 0\ \exists\ n_0 \in \mathbb{N} \ni$

$$\rho(x_n, x_m) = \sup\{|x_i^{(n)} - x_i^{(m)}| : i \in \mathbb{N}\} < \varepsilon \quad \forall\ n, m \geq n_0$$

where $x_n = x_i^{(n)} = x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots$

$\Rightarrow |x_i^{(n)} - x_i^{(m)}| < \varepsilon \quad \forall\ n, m \geq n_0$ and $\forall$ fixed i --- (i)

$\Rightarrow (x_i^{(n)})$ is Cauchy in $\mathbb{R}$ and so it converges to a point say $x = (x_i)$ in $\mathbb{R}$. i.e.

$$\lim_{n\to\infty} x_i^{(n)} = x_i \quad \forall \text{ fixed } i.$$

Now letting $m \to \infty$ in (i) we have

$$|x_i^{(n)} - x_i| < \varepsilon \quad \forall\ n \geq n_0$$

$\Rightarrow \sup\{|x_i^{(n)} - x_i| : i \in \mathbb{N}\} < \varepsilon, \ \forall\ n \geq n_0$

$\Rightarrow \rho(x_n, x) < \varepsilon, \ \forall\ n \geq n_0$

$\Rightarrow x_n \to x = x_i$ as $n \to \infty$

$\therefore \{x_n\}$ converges to x.

$\square$

It remains to show that $x = (x_i)$ is in C i.e a ~~constant~~ convergent sequence. Since $(x_i^{(n_0)})$ is a convergent sequence in $\mathbb{R}$ then it is Cauchy i.e $\exists\ i_0 \in \mathbb{N} \ni |x_i^{(n)} - x_j^{(n)}| < \varepsilon/3 \ \forall\ i, j \geq i_0$

Now $|x_i - x_j| = |x_i - x_i^{(n_0)} + x_i^{(n_0)} - x_j^{(n_0)} + x_j^{(n_0)} - x_j|$

$$\leq |x_i - x_i^{(n_0)}| + |x_i^{(n_0)} - x_j^{(n_0)}| + |x_j^{(n_0)} - x_j|$$

$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

i.e $|x_i - x_j| < \varepsilon \ \forall\ i, j \geq i_0$

$\Rightarrow (x_i)$ is Cauchy in $\mathbb{R}$ and it converges in $\mathbb{R}$.

~~Hence~~ $x = (x_i)$ is in C.

Thus (C, ρ) is a Complete metric space. $\square$

Example 3 The metric space (C_0, ρ) with the metric ρ defined by $\rho(x, y) = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$ is a Complete metric space, where $x = x_0$; $y = y_0$ in C_0.

C.A TEST on 26/5/21

23/3/2020

After corona virus Break 19/4/21

Ahlfors pg 33 or 47

20/4/2021

Continous Mapping on metric Space

DEFINITION: Let $f:(X,\rho)\longrightarrow(Y,\gamma)$ be a mapping from a matric space (X,ρ) to another metric space (Y,γ). Then f is said to be continuous at a point $x_0\in X$ if $\forall$ $\varepsilon>0$ $\exists$ $\delta>0$ $\ni$ $\rho(x,x_0)<\delta$
$\Rightarrow \gamma(f(x),f(x_0))<\varepsilon$.

If f is continous at every point of X, then f is said to be continous on X.

REMARK: Now recall the defination of an open ball in a metric space (X,ρ) the set

$$B_r(x)=\{x_0\in(X,\rho):\rho(x,x_0)<r\}$$ is called

the open ball centred at x_0 and radius $r>0$.
Now using ε-δ defination and a mapping f from a metric space (X,ρ) to a metric space (Y,γ) We define an open ball in (X,ρ) and in (Y,γ) as follow

$B_\delta(x_0)=\{x\in X:\rho(x,x_0)<\delta\}$ is an open set in (X,ρ).

$B_\varepsilon(x_0)=\{f(x)\in Y:\gamma(f(x),f(x_0))<\varepsilon\}$.

LEMMA 2.3: Let $f:(X,\rho)\longrightarrow(Y,\gamma)$ be a mapping from a metric space (X,ρ) in a metric space (Y,γ). Then f is continous at a point $x_0\in X$ iff for any open ball $B_\varepsilon(f(x))$ in Y, $\exists$ an open ball
$B_\delta(x_0)\in X \ni f(B_\delta(x_0))\subseteq B_\varepsilon(f(x_0))$.

proof:

Suppose f is continous at a point $x_0 \in X$ then by defination, $\forall\, \varepsilon > 0\ \exists\ \delta > 0 \ni \rho(x, x_0) < \delta$

$\Rightarrow \gamma(f(x), f(x_0)) < \varepsilon$

$\Rightarrow\quad x \in B_\delta(x_0) \Rightarrow f(x) \in B_\varepsilon(f(x_0))$

$\Leftrightarrow f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0))$ or

$B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(F(x_0))$

$f(B_r(x_0) \subseteq B_\varepsilon(f(x_0))$ $\square$

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LEMMA 2.4: Let $(x, \rho) \longrightarrow (Y, \gamma)$ be a mapping from a metric space (X, ρ) to a metric space (Y, γ). Then f is continuous on X iff $\forall$ open set U in Y $f^{-1}(u)$ is an open set in X.

proof: Suppose f is continous and let U be an open set in Y. Then we show that $f^{-1}(U)$ is an open set in X.

To see this let $x_0 \in f^{-1}(u)$ then $f(x_0) \in U$

Since U is open then $\exists\ \epsilon > 0 \ni B_\varepsilon(f(x_0)) \subset U$

Since f is continuous and $x_0 \in B_\varepsilon(f(x_0))$ then

$\exists\ \delta > 0 \ni f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0)) \subset U$ (By the above lemma)

$\Rightarrow\quad B_\delta(x_0) \subset f^{-1}(u)$

Since $x_0 \in f^{-1}(u)$ is arbitrary the $f^{-1}(u)$ is an open set in X.

Conversely, suppose that $\forall$ open set U in Y, $f^{-1}(U)$ is open in X. then we shall show that f is continuous. Now let $x_0 \in X$ be arbitrary and let $\varepsilon > 0$ be given then $B_\epsilon(f(x_0))$ is an open ball in Y and so by the hypothesis $f^{-1}(B_\epsilon(f(x_0)))$ is an open set in Y. since $x_0 \in f^{-1}(B_\epsilon(f(x_0)))$ then $\exists\ \delta > 0 \ni B_\delta(x_0) \subset f^{-1}(B_\epsilon(f(x_0)))$

$\Rightarrow f(B_\delta(x_0)) \subset B_\epsilon(f(x_0))$ [By The above Lemma]

$\Rightarrow f$ is continuous at x_0 in X ■

REMARK: suppose $x_n \to x_0$ as $n \to \infty$ but f is not continuous at x_0 $\forall\ \delta > 0\ \exists\ \epsilon > 0 \ni \rho(x, x_0) < \delta$

$\Rightarrow \gamma(f(x), f(x_0)) \geq \epsilon$.

LEMMA 2.5: Let $f : (X, \rho) \to (Y, \gamma)$ be a mapping from a metric space (X, ρ) to a metric space (Y, γ) then f is continuous at $x_0 \in X$ iff $\forall\ (x_n) \subset X \ni x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$.

proof:

suppose f is continuous at $x_0 \in X$. Then by defn $\forall\ \epsilon > 0\ \exists\ \delta > 0 \ni \rho(x, x_0) < \delta$

$\Rightarrow \gamma(f(x), f(x_0)) < \epsilon$ ——— (1)

Suppose $x_n \xrightarrow[n\to\infty]{} x_0$. Then by defn $\forall\ \delta > 0\ \exists\ n_0 \in \mathbb{N} \ni \rho(x_n, x_0) < \delta\ \forall\ n \geq n_0$ ——— (2)

Now putting (2) into (1) we see that

$\rho(x_n, x_0) < \delta \Rightarrow \gamma(f(x_n), f(x_0)) < \epsilon \; \forall \; n \geq n_0$.

$\Rightarrow f(x_n) \xrightarrow[n \to \infty]{} f(x_0)$

Conversely, suppose that $\forall$ sequence (x_n) in X $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$. Then we show that f is continuous at x_0. Suppose f is not continuous at x_0. Let $\delta = 1$ and $x_1 \in X$ be such that $\rho(x_1, x_0) < \delta_1 \Rightarrow \gamma(f(x_1), f(x_0)) \geq \epsilon$ and $\delta_2 = \frac{1}{2}$ and $x_2 \in X$ be such that

$\rho(x_2, x_0) < \delta_2 \Rightarrow \gamma(f(x_2), f(x_0)) \geq \epsilon$ and

$\delta_3 = \frac{1}{3}$ and $x_3 \in X$ be such that

$\rho(x_2, x_0) < \delta_3 \Rightarrow \gamma(f(x_3), f(x_0)) \geq \epsilon$.

So continuing in this way for any

$\delta_n = \frac{1}{n} \; \exists \; (x_n) \subset X \ni x_n \xrightarrow[n \to \infty]{} x_0$

but $f(x_n) \not\to f(x_0)$. This contradicts the fact that $x_n \to x_0$

$\Rightarrow f(x_n) \to f(x_0)$.

Hence the assumption that f is not continuous is not true.

Hence, f is continuous $\square$

LEMMA 2.6: Let $f(X,\rho) \to (Y,\gamma)$ be a mapping from a metric space (X,ρ) to a metric space (Y,γ), then f is continuous iff $\forall$ closed set F in Y, $f^{-1}(F)$ is a closed set in X.

<u>proof:</u>

Suppose f is continuous and let F be any closed set in Y. Then F^c is open set in Y and so $f^{-1}(F^c)^c = f^{-1}(F)$ which is closed set in X.

Conversely, suppose that for any closed set F in Y, $f^{-1}(F)$ is a closed set in X. Since F is a closed set in Y, then F^c is an open set in Y and since $f^{-1}(F)$ is a closed set in X then $(f^{-1}(F))^c = f^{-1}(F^c)$ is an open set in X and f is continuous since inverse image of open set in Y is open set in X $\square$

DEFINATION: Let (X,ρ) be any metric space and let $\{U_\alpha : \alpha \in I\}$ where I is an index set, be a family of subsets of X. Then the family is said to cover X (or said to be a covering of X) if

$$X \subseteq \bigcup_{\alpha \in I} U_\alpha$$

Now, if for each $\alpha \in I$, U_α is open and the family $\{U_\alpha : \alpha \in I\}$ covers X (i.e $X \subseteq \bigcup_{\alpha \in I} U_\alpha$ and each V_α is open) then we say that the family is open covering for X.

Now the family $\{V_\alpha : \alpha \in I\}$ is a sub-family of $\{U_\alpha : \alpha \in I\}$ and $X \subseteq \bigcup_{\alpha \in I} V_\alpha$ then we say that the sub-family $\{V_\alpha : \alpha \in I\}$ covers X. i.e the sub-family $\{V_\alpha : \alpha \in I\}$ that covers X is called <u>sub covering</u>.

DEFINATION: ~~A~~ metric space (X, ρ) is said to be compact, if every open covering of X has a finite sub-covering i.e X is compact if for every open family $\{U_\alpha, \alpha \in I\}$ that covers X (i.e $X \subseteq \bigcup_{\alpha \in I} U_\alpha$) there exist a finite sub-family $(U_{\alpha_i} : i = 1, 2, \ldots n\} \ni X \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$.

DEFINATION: i) A metric space (X, ρ) is said to have Bolzano-weierstrass property (BWP) if every infinite subset of X has a limit point.

ii) ~~A~~ metric space (X, ρ) is said to be sequentially compact if every sequence in X has a convergent ~~sub~~ sub-sequence.

iii) ~~A metric space (X,ρ) is said to be totally bounded if it has finite ϵ-net~~ subset i.e a metric space (X,ρ) is totally bounded if $\forall\ \epsilon>0$, $\exists\ x_1, x_2, \dots, x_k \in X$ with k finite $\ni$

$\{B_\epsilon(x_i) : 1 \le i \le k\}$ is an open Cover in X.

iv) ~~A metric space (X,ρ) is said to have finite Intersection property (FIP)~~ if every family of subsets of X has a finite subfamily with non-empty Intersection; i.e. (X,ρ) has finite Intersection property if $\{F_\alpha : \alpha\in I\}$ is a family of subsets of X $\exists$ a subfamily $\{F_{\alpha_i} : 1,2,\dots n\}$

$$\ni \bigcap_{i=1}^{n} F_{\alpha_i} \neq \phi.$$

5/5/21

PROPERTIES OF COMPACTNESS

i Every Compact metric space has a BOLZANO WEIERSTRASS property (BWP).

<u>Proof:</u>

Let (x,ρ) be a compact metric space; and let A be an Infinite subset of X. Suppose A has no limit point.

then $\forall x \in A \; \exists \; r > 0 \ni B_r(x) \cap A = \phi$ since every open ball is an open set. Then, the family $\{B_r(x) : x \in X\}$ forms an open covering of X. Let $X \subseteq \bigcup B_r(x)$ since X is compact then $\exists \; x \in X$ finite points $x_1, x_2, --- x_n \ni X \subseteq \bigcup_{i=1}^{n} B_r(x_i)$.

Since $A \subseteq X$ and $X \subseteq \bigcup_{i=1}^{n} B_r(x_i)$ then $A \subseteq \bigcup_{i=1}^{n} B_r(x_i)$. But this shows that A is finite. But this contradicts the hypothesis that A is infinite and so the assumption that A has no limit point is false hence, A must have a limit point.

Thus A has a BWP ∎

ii) Every closed subset of compact metric space is compact.

__Proof:__

Let A be any closed subset of a metric space (X, ρ). Then we shall show that A is compact. To see this, let $\{U_\alpha : \alpha \in I\}$ be an open covering of A. Since A is closed then A^c is open and so

$X = A \cup A^c$ since $A \subseteq \bigcup_{\alpha \in I} U_\alpha$. then

$$X \subseteq \left(\bigcup_{\alpha \in I} U_\alpha\right) \cup A^c$$

This shows that the family $\{U_\alpha : \alpha \in I\} \cup A^c$ forms an open covering of X since X is compact then $\exists\ \alpha_1, \alpha_2, \dots \alpha_n$ i.e. $\exists\ U_{\alpha_1}, U_{\alpha_2}, \dots U_{\alpha_n} \ni$

$$X \subseteq \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) \cup A^c$$

$$\Rightarrow A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

Hence ~~A~~ is compact. $\square$

iii) Every sequentially compact metric space is totally bounded.

Proof:

Let (X, ρ) be a sequentially compact metric space. Suppose X is not totally bounded. Then X has no finite ϵ-net subset. so for $x_1 \in X$, $\exists$ $x_2 \in X \ni \rho(x_1, x_2) \geq \epsilon$, otherwise $\{x_1\}$ would be a finite ϵ-net subset of X. Again $\exists\ x_3 \in X \ni \rho(x_1, x_3) \geq \epsilon$ and $\rho(x_2, x_3) \geq \epsilon$ otherwise $\{x_1, x_2\}$ would be a finite ϵ-net subset. Continuing in this way, we get a sequence $\{x_n\}$ such that

$$\rho(x_n, x_m) \geq \epsilon \text{ for } m \neq n.$$

$\Rightarrow \{x_n\}$ cannot contain a convergent subsequence. But this contradicts the sequentially compactness of X. Hence, X must be totally bounded $\square$

iv) A metric space (X, ρ) is compact iff for every closed subset of X with finite intersection property (F.I.P) has non-empty intersection.

proof:-

Let (X, ρ) be a metric space and let $\{U_\alpha : \alpha \in I\}$ be a family of closed subsets of X with finite intersection property.

Suppose $\bigcap_{\alpha \in I} U_\alpha = \phi$

$\Rightarrow \bigcup_{\alpha \in I} U^c = \phi^c = X$. Since for $\alpha \in I$, U_α is closed then for each $\alpha \in I$, U_α^c is open and so $\{U_\alpha^c : \alpha \in I\}$ form an open covering of X. Since X is compact $\exists\ U_{\alpha_1}, U_{\alpha_2} \dots U_{\alpha_n}$ $\ni X \subseteq \bigcup_{i=1}^{n} U_\alpha^c \Rightarrow \phi = \bigcap_{i=1}^{n} U_\alpha$

But this contradicts the F.I.P of $\{U_\alpha : \alpha \in I\}$

Hence, the assumption that

$$\bigcap_{i=1}^{n} U_\alpha = \phi \text{ is false}$$

Hence $\bigcap_{\alpha \in I} U_\alpha \neq \phi$.

Conversely, suppose that the family $\{V_\alpha : \alpha \in I\}$ of closed subsets of X with F.I.P has non-empty intersection. Then we want to show that X is compact. To see this, let $\{F_\alpha : \alpha \in I\}$ be an open covering of X. Suppose that $X = \bigcup_{\alpha \in I} F_\alpha \Rightarrow \emptyset = X^c = \bigcap_{\alpha \in I} F_\alpha^c$. Since for each $\alpha \in I$, F_α is an open set then for each $\alpha \in I$, F_α^c is closed i.e $\{F_\alpha^c : \alpha \in I\}$ is a family of closed sets with

$(\cap A)^c = \cup A^c$

$$\bigcap_{\alpha \in I} F_\alpha^c = \emptyset.$$

But this contradicts the hypothesis and so $\exists\ F_{\alpha_1}^c, F_{\alpha_2}^c, \ldots, F_{\alpha_n}^c \ni \bigcap_{i=1}^{n} F_{\alpha_i}^c = \emptyset$

$$\Rightarrow X = \bigcup_{i=1}^{n} F_{\alpha_i}.$$

Hence, X is compact $\square$

V) A continuous image of compact metric space is compact. i.e $f : X \to Y$ is a continuous mapping on a compact metric space X to another metric space Y.

Then $f(X)$ is a compact subset of Y.

proof:

To show that $f(X)$ is a compact subset of Y, let $\{U_\alpha : \alpha \in I\}$ be an open covering of $f(X)$.

i.e. $f(X) \subseteq \bigcup_{\alpha \in I} U_\alpha$

$$\Rightarrow X \subseteq f^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(U_\alpha)$$

Since for each $\alpha \in I$, U_α is an open set in Y and f is continuous then $f^{-1}(U_\alpha)$ is an open set in X.

So, the family $\{f^{-1}(U_\alpha) : \alpha \in I\}$ form an open covering of X. Since X is compact $\exists\, f^{-1}(U_{\alpha_1}), f^{-1}(U_{\alpha_2}), \ldots \ldots f^{-1}(U_{\alpha_n}) \ni$

$$X \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i}) = f^{-1}\left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)$$

$$\Rightarrow f(X) = \bigcup_{i=1}^{n} U_{\alpha_i}$$

$\Rightarrow f(X)$ is compact. $\square$

vi) A metric space is said to be sequentially compact iff it has a Bolzano-Weierstrass property. ($SC \Leftrightarrow BWP$).

proof: $SC \Rightarrow B.W.P$

Let (X,ρ) be a sequentially compact metric space. Then we want to show that it has a BWP. Let A be an infinite subset of X and let (x_n) be a sequence in A. Since $A \subset X$. then (x_n) is ~~X~~. Since X is sequentially compact then it has a subsequence $(x_{n_k}) \ni$

$x_{n_k} \xrightarrow[n\to\infty]{} x$ i.e $\forall\ \epsilon > 0\ \exists\ K_0 \in \mathbb{N} \ni$

$\rho(x_{n_k}, x) < \epsilon\ \forall\ k \geq K_0$

$\Rightarrow x_{n_k} \in B_\epsilon(x)\ \forall\ k \geq K_1$

$\Rightarrow x_{n_k} \in B_\epsilon(x)$ But $x_{n_{k_0}} \in A$ and so

$B_\epsilon(x) \cap A \neq \emptyset$

$\Rightarrow x$ is a limit point of A.

Hence, X has a B.W.P.

BWP $\Rightarrow$ SC

Let (X,ρ) be a metric space with BWP then we want to show that X is S.C. Let (x_n) be a sequence in X. Now, if $A = \{x_n : n \in \mathbb{N}\}$ is its range and A is infinite then it has a limit point say x, so $\forall\ \epsilon > 0$, we have that

$B_r(x) \cap A \neq \emptyset$

$\Rightarrow B_r(x) \cap A$ has infinitely many points.

Choose $x_{n_1} \in B_1(x) \cap A$,
$x_{n_2} \in B_{1/2}(x) \cap A$, ... $x_{n_k} \in B_{1/k}(x) \cap A$

$\Rightarrow \rho(x_{n_k}, x) < \frac{1}{k} \xrightarrow[n \to \infty]{} 0$

$\Rightarrow x_{n_k} \xrightarrow[k \to \infty]{} x$

$\Rightarrow x$ is sequentially compact.